STABILITY INDEX FOR CHAOTICALLY DRIVEN CONCAVE MAPS

GERHARD KELLER

ABSTRACT. We study skew product systems driven by a hyperbolic base map $\hat{S}:\Theta\to\Theta$ (e.g. a baker map or an Anosov surface diffeomorphism) and with simple concave fibre maps on $\mathbb{R}_+$ like $x\mapsto\hat{g}(\theta)\arctan(x)$ where $\theta\in\Theta$ is a parameter driven by the base map. The fibre-wise attractor is the graph of an upper semicontinuous function $\theta\mapsto\hat{\varphi}_\infty(\theta)\in\mathbb{R}_+$. For many choices of $\hat{g}$, $\hat{\varphi}_\infty$ has a residual set of zeros but $\hat{\varphi}_\infty>0$ μ_{SRB}-a.s. where μ_{SRB} is the Sinai-Ruelle-Bowen measure of $\hat{S}^{-1}$.

In such situations we evaluate the stability index of the global attractor of the system, which is the subgraph $\{(\theta,x)\in\Theta\times\mathbb{R}_+ : 0\leqslant x\leqslant\hat{\varphi}_\infty(\theta)\}$ of $\hat{\varphi}_\infty$, at all regular points $(\theta,0)$ in terms of the local exponents $\hat{\Gamma}(\theta):=\lim_{n\to\infty}\frac{1}{n}\log\hat{g}_n(\theta)$ and $\hat{\Lambda}(\theta):=\lim_{n\to\infty}\frac{1}{n}\log|D_u\hat{S}^{-n}(\theta)|$ and of the positive zero s_* of a certain thermodynamic pressure function associated with $\hat{S}$ and $\hat{g}$. (In queuing theory, an analogon of s_* is known as Loyne's exponent [12].)

The stability index was introduced by Podvigina and Ashwin [16] to quantify the local scaling of basins of attraction.

1. Introduction

1.1. **Motivation.** Consider a monotone concave map h that maps some interval $[0,a]$ into itself with $h(0)=0$ and $h'(0)=1$. The family $h_r(x)=rh(x)$ with $0\leqslant r\leqslant h(a)^{-1}$ has a very simple bifurcation scenario: for $r\leqslant 1$, the point 0 is a globally attracting fix point, that looses its stability at $r=1$ and gives birth to a new stable fixed point $x_s>0$ which attracts all points except the fixed point 0.

If the bifurcation parameter r is not fixed but is driven by some ergodic dynamics, the scenario becomes a bit more complex. Quasiperiodic drives may lead to the creation of strange non-chaotic attractors (SNA) as the result of the loss of stability of a stable non-autonomous fixed point, a phenomenon that attracted much attention both in the physics and the mathematics literature, see e.g. the references collected in [4, 6]. More recently, also systems with chaotic drives were studied - mostly in the physics literature where they are used as simple examples to study generalized synchronisation, see e.g. [19]. Due to the presence of many different normal Lyapunov exponents associated to different invariant measures of the chaotic driving system, the loss of stability of the globally attracting non-autonomous fixed point at 0 and the creation of an attracting non-autonomous fixed point which is everywhere strictly positive is a complicated process that happens while the parameter varies in a nontrivial

Date: September 12, 2012.

2010 *Mathematics Subject Classification.* 37D20, 37D45, 37G35, 37H20.

Key words and phrases. Stability index, skew product, strange invariant graph.

This work is funded by DFG grant Ke 514/8-1. I am indebted to A. Otani (Erlangen) for many helpful remarks on this paper and for producing Figure 1, and I thank R. Ramaswamy and his group (University of Hyderabad, India) for their hospitality during a visit in March 2012, where a huge part of this research was done.

interval [19]. The goal of this paper is to describe some quantitative features of this process in simple model situations.

1.2. **The class of systems.** We study skew product systems where the driving system is a bijective bi-measurable map $\hat{S} : \Theta \to \Theta$ on a measurable space $(\Theta, \mathcal{A})$ that has good hyperbolicity properties to be specified below. The fibre maps from an interval $I := [0, a]$ into itself are of the form $x \mapsto \hat{g}(\theta)h(x)$ where $\hat{g} : \Theta \to (0, \infty)$ and $h : I \to \mathbb{R}_+$ is a strictly increasing, concave C^{1+}-function with $h(0) = 0$ and $h'(0) = 1$.[1] Let $\Omega = \Theta \times I$. Then the driven system is described by

$$F : \Omega \to \Omega, \quad F(\theta, x) = (\hat{S}\theta, \hat{g}(\theta)h(x)) . \tag{1.1}$$

Denote by $F^n_\theta : I \to I$ the fibre map of the iterated map F^n, i.e. $F^n_\theta(x)$ is the second component of $F^n(\theta, x)$.

The global pullback attractor of this system is the set

$$\{(\theta, x) \in \Omega : 0 \leqslant x \leqslant \hat{\varphi}_\infty(\theta)\} \tag{1.2}$$

where $\hat{\varphi}_\infty : \Theta \to I$ is the *maximal invariant graph* (with the slight abuse of terminology that we do not distinguish between the function and its graph). It is defined for all $\theta \in \Theta$ by

$$\hat{\varphi}_\infty(\theta) = \lim_{n\to\infty} \hat{\varphi}_n(\theta), \text{ where } \quad \hat{\varphi}_n(\theta) := F^n_{\hat{S}^{-n}\theta}(a) . \tag{1.3}$$

The limit exists and is measurable, because $\hat{\varphi}_{n+1}(\theta) = F^n_{\hat{S}^{-n}\theta}(F_{\hat{S}^{-(n+1)}\theta}(a)) \leqslant F^n_{\hat{S}^{-n}\theta}(a) = \hat{\varphi}_n(\theta)$ in view of the monotonicity of the fibre maps. If Θ is a topological space and if all $\hat{g} \circ \hat{S}^{-n}$ are continuous, then also all $\hat{\varphi}_n$ are continuous so that $\hat{\varphi}_\infty$ is upper semicontinuous.

In order to obtain some quantitative, dimension-like information about $\hat{\varphi}_\infty$, we need some additional uniformly hyperbolic or expanding structure for the system. The following assumptions are a compromise between the goal to cover a number of different examples and to keep technicalities at a moderate level.

Hypothesis 1. There is a piecewise expanding and piecewise C^{1+} mixing Markov map $S : \mathbb{T}^1 \to \mathbb{T}^1$ with finitely many branches which is a factor of $\hat{S}^{-1}$, i.e.

$$S \circ \Pi = \Pi \circ \hat{S}^{-1} \quad \text{for some measurable} \quad \Pi : \Theta \to \mathbb{T}^1 . \tag{1.4}$$

It is a well known fact that S has a unique invariant probability measure μ_{ac} absolutely continuous w.r.t. Lebesgue measure m on $\mathbb{T}^1$.

Remark 1. One can also admit countable Markov maps with finite range structure, and a careful look at the proofs reveals possibilities to weaken the assumption on S even further.

Hypothesis 2. The multiplier function $\hat{g}$ depends only on $\Pi\theta$, i.e.

$$\hat{g}(\theta) = g(\Pi\theta) \tag{1.5}$$

for a suitable function $g : \mathbb{T}^1 \to (0, \infty)$. (How to deal with more general multiplier functions when $\hat{S}$ is (piecewise) hyperbolic, is explained in Remark 4.) Let $g_n = \prod_{i=1}^n g \circ S^i$, and denote by $\mathcal{U}_n(v)$ the family of all interval neighbourhoods U of $v \in \mathbb{T}^1$ such that $S^n_{|U} : U \to S^n U$ is a diffeomorphism. We assume that the family of all $g_{n|U}$ with $n \geqslant 1$, $v \in \mathbb{T}^1$ and $U \in \mathcal{U}_n(v)$

[1] Here and in the sequel C^{1+} means "C^1 with Hölder continuous derivative" without specifying the Hölder exponent.

has uniformly bounded distortion in the following sense: There is a constant $D > 0$ such that for all $n > 0$, all $v \in \mathbb{T}^1$, all $U \in \mathcal{U}_n(v)$ and all $\tilde{v} \in U$

$$D^{-1} \leqslant \left| \frac{g_n(\tilde{v})}{g_n(v)} \right| \leqslant D \,. \tag{1.6}$$

Remark 2. If $\log g$ is Hölder continuous on each monotonicity interval of S, assumption (1.6) is a simple classical consequence of the uniform expansion of S. Similarly we have (enlarging D, if necessary)

$$D^{-1} \leqslant \left| \frac{(S^n)'(\tilde{v})}{(S^n)'(v)} \right| \leqslant D \,. \tag{1.7}$$

Remark 3. The variable θ enters the definition of the approximating functions $\hat{\varphi}_n$ only via the values $\hat{g}(\hat{S}^{-k}\theta) = g(S^k(\Pi\theta))$, $k = 1, \dots, n$. Therefore the graph $\hat{\varphi}_\infty(\theta)$ depends on θ only via $\Pi\theta$ so that there is a measurable function $\varphi_\infty : \mathbb{T}^1 \to I$ such that $\hat{\varphi}_\infty(\theta) = \varphi_\infty(\Pi\theta)$. The geometric properties of this function are what we are basically interested in. Corresponding properties of the function $\hat{\varphi}_\infty$ will follow as corollaries.

The following is a well known consequence of the semi-uniform ergodic theorem [20] and of the uniform concavity of the fibre maps: $\varphi_\infty(v) = 0$ for all $v \in \mathbb{T}^1$ if $\int_{\mathbb{T}^1} \log g \, d\mu < 0$ for all S-invariant probability measures μ, and φ_∞ is strictly positive if $\int_{\mathbb{T}^1} \log g \, d\mu > 0$ for all such μ. The most interesting situation occurs under the following hypothesis:

Hypothesis 3. There is an S-invariant probability measures μ_- such that

$$\int \log g \, d\mu_- < 0 < \int \log g \, d\mu_{\mathrm{ac}} \,. \tag{1.8}$$

Note that under this assumption $\log g$ is not cohomologous to a constant and that it is easy to prove (see [7, 9]) that $\varphi_\infty(v) > 0$ for μ_{ac}-a.e. v.

Example 1 (**Baker transformations**)**.** Let $\Theta = [0, 1)^2$ and let $\hat{S} : \Theta \to \Theta$ be a baker transformation

$$\hat{S}(u, v) = \begin{cases} \big(s^{-1}u, sv\big) & \text{if } u < s \\ \big((1-s)^{-1}(u-s), s + (1-s)v\big) & \text{if } u \geqslant s \,. \end{cases} \tag{1.9}$$

With $\Pi(u, v) = v$ and with $S(v) = s^{-1}v$ for $v < s$ and $S(v) = (1 - s)^{-1}(v - s)$ if $v \geqslant s$ this fits the above setting. Figure 1 shows plots of the invariant graph $\varphi_\infty(v)$ when $s = 0.45$, $h(x) = \arctan(x)$ and the multiplier function $g : \mathbb{T}^1 \to (0, \infty)$ is $g(v) = r \cdot (1 + \epsilon + \cos(2\pi v))$ with $\epsilon = 0.01$. Observe that in this example all $\hat{g} \circ \hat{S}^{-n}$ are continuous when interpreted as defined on the circle $\mathbb{T}^1$ so that $\hat{\varphi}_\infty$ and φ_∞ are upper semicontinuous. Our main results shed some light on the structure of φ_∞ close to the base line, i.e. when these values are small.

In this example, the S-invariant measure δ_0 maximizes $\int \log g \, d\mu$ (the value is $\log(r \cdot 2.01)$), and the equidistribution on the period-3 orbit $[0.10255, 0.22788, 0.50640]$ apparently minimizes this quantity (the value is $\log(r \cdot 0.28216)$). The corresponding value for Lebesgue measure $\mu = m$ is $\log(r \cdot 0.57589)$. So assumption (1.8) is satisfied for parameters $r \in [0.57589^{-1}, 0.28216^{-1}] = [1.7364, 3.5441]$, and the parameters used in Figure 1 are in this range.

Remark 4. Baker transformations are particularly simple examples where the sets $\Pi^{-1}(v)$ are uniformly stable fibres for the action of $\hat{S}^{-1}$ on Θ. In such situations one can also deal with multiplier functions $\hat{g}(\theta)$ that do not only depend on $\Pi\theta$ as required in Hypothesis 2.

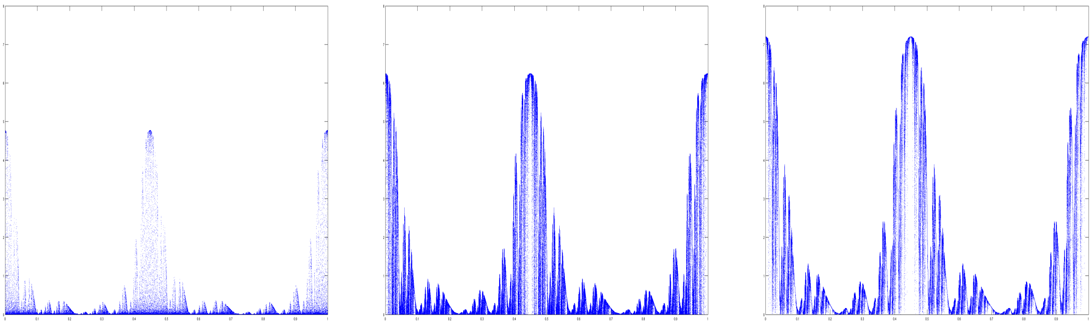

FIGURE 1. The graph $\varphi_\infty(v)$ for the baker map from Example 1. The parameters are (from left to right) $r = 1.74$, $r = 2.2$, $r = 2.5$.

Under suitable assumptions, a classical construction which goes back to works of Sinai and of Bowen yields functions $\hat{b} : \Theta \to \mathbb{R}$ and $g : \mathbb{T}^1 \to (0, \infty)$ such that

$$\log \hat{g}(\theta) = \log g(\Pi\theta) + \hat{b}(\theta) - \hat{b}(\hat{S}^{-1}\theta)\,. \tag{1.10}$$

More precisely, we assume:

i) $\log \hat{g} : \Theta \to \mathbb{R}$ is Hölder continuous. (Hölder continuity on each set $\Pi^{-1}J$ where J is a monotonicity interval of S suffices.)
ii) There is an injection $\varsigma : \mathbb{T}^1 \to \Theta$ which is Hölder continuous on monotonicity intervals of S, which satisfies $\Pi \circ \varsigma = \mathrm{id}_{\mathbb{T}^1}$, and which is such that each $\theta \in \Theta$ belongs to the stable fibre of $\varsigma\Pi\theta$ in the sense that

$$\exists C > 0\ \exists r \in (0,1)\ \forall \theta \in \Theta\ \forall n > 0 :\ d(\hat{S}^{-n}\theta, \hat{S}^{-n}(\varsigma\Pi\theta)) \leqslant C\, r^n\,. \tag{1.11}$$

Following [2, Lemma 1.6], define

$$\hat{b}(\theta) = \sum_{n=0}^{\infty} \Big(\log \hat{g}(\hat{S}^{-n}\theta) - \log \hat{g}(\hat{S}^{-n}\varsigma\Pi\theta) \Big)\,. \tag{1.12}$$

As $\log \hat{g}$ is Hölder continuous, $\|\hat{b}\|_\infty := \sup_{\theta\in\Theta} |\hat{b}(\theta)| < \infty$, and

$$\hat{b}(\theta) - \hat{b}(\hat{S}^{-1}\theta) = \log \hat{g}(\theta) - \left[\log \hat{g}(\varsigma\Pi\theta) + \sum_{n=1}^{\infty} \Big(\log \hat{g}(\hat{S}^{-n}\varsigma\Pi\theta) - \log \hat{g}(\hat{S}^{-n+1}\varsigma\Pi\hat{S}^{-1}\theta) \Big) \right].$$

The term in brackets depends only on $\varsigma\Pi\theta$, and we denote it by $\log g(\Pi\theta)$. Then

$$\hat{b}(\theta) - \hat{b}(\hat{S}^{-1}\theta) = \log \hat{g}(\theta) - \log g(\Pi\theta)\,, \tag{1.13}$$

and one can show that $\hat{b}$ and $\log \hat{g}$ are Hölder continuous [2, Lemma 1.6]. In particular, the distortion bounds of Hypothesis 2 are satisfied.

Denote now by $\hat{\varphi}_\infty$ the invariant graph of the system with multiplier $\hat{g}$, and by $\varphi_\infty \circ \Pi$ the invariant graph of the system with multiplier $g \circ \Pi$. We prove the following proposition in section 6.

Proposition 1. *For each $\theta \in \Theta$, $\hat{\varphi}_\infty(\theta) > 0$ if and only if $\varphi_\infty(\Pi\theta) > 0$, and if this is the case, then*
$$|\log \hat{\varphi}_\infty(\theta) - \log \varphi_\infty(\Pi\theta)| \leqslant \log \frac{a}{h(a)} + 2\|\hat{b}\|_\infty . \tag{1.14}$$

Example 2 (Anosov surface diffeomorphism). Let $\Theta = \mathbb{T}^2$ and let $\hat{S} : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^2 Anosov diffeomorphism. It has a Markov partition $\{R_1, \dots, R_p\}$ [18]. As indicated in the proof of Lemma 3 in [17] (see also section 6.3) one can construct a C^{1+} expanding Markov interval map $S : \mathbb{T}^1 \to \mathbb{T}^1$ that is a factor of $\hat{S}^{-1}$, i.e. $S \circ \Pi = \Pi \circ \hat{S}^{-1}$ with the projection $\Pi : \mathbb{T}^2 \to \mathbb{T}^1$ and the injection $\varsigma : \mathbb{T}^1 \to \mathbb{T}^2$ defined in section 6.3. If $\hat{g} : \mathbb{T}^2 \to (0, \infty)$ is a Hölder function, then there are functions $g : \mathbb{T}^1 \to (0, \infty)$ and $\hat{b} : \mathbb{T}^2 \to \mathbb{R}$ such that $\log \hat{g} = \log g \circ \Pi + \hat{b} - \hat{b} \circ \hat{S}^{-1}$ and $\log g$ is Hölder continuous on every monotonicity interval of S, compare Remark 4.

Denote by $\hat{\mu}^-_{\text{SRB}}$ the SRB-measure of $\hat{S}^{-1}$. It projects to a S-invariant measure μ^-_{SRB} on $\mathbb{T}^1$. As $\hat{\mu}^-_{\text{SRB}}$ is absolutely continuous on unstable fibres of $\hat{S}^{-1}$, the measure $\mu^-_{\text{SRB}} \circ \Pi^{-1}$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{T}^1$, so that it coincides with the unique absolutely continuous invariant measure μ_{ac} of S from Hypothesis 1. Using the explicit representation for the Jacobian $D\Pi$ of the holonomy along stable fibres of $\hat{S}^{-1}$ (in this case: the absolute value of the derivative of the holonomy), it is not hard to prove that
$$\log |D_u \hat{S}^{-1}(\theta)| = \log |S'(\Pi\theta)| + \log D\Pi(\theta) - \log D\Pi(\hat{S}^{-1}\theta) . \tag{1.15}$$
For completeness the proof is provided in section 6. Here D_u denotes the derivative in the unstable direction of $\hat{S}^{-1}$. Proposition 1 applies in this situation so that the graphs of $\hat{\varphi}_\infty$ and of $\varphi_\infty \circ \Pi$ can again be compared as in (1.14).

2. Main results

Throughout we assume that Hypotheses 1 - 3 are satisfied.

2.1. Global scaling properties.

A global characteristic of the invariant graph $\varphi_\infty : \mathbb{T}^1 \to [0, \infty)$ is the distribution of its values - in particular of values close to zero - under Lebesgue measure m. Recall that $\varphi_\infty(v) > 0$ for m-a.e. $v \in \mathbb{T}^1$ by Hypothesis 3.

For $s \in \mathbb{R}$ denote by $\mathcal{L}_s$ the transfer operator
$$\mathcal{L}_s : L^1_m \to L^1_m,\ \mathcal{L}_s f(v) = \sum_{\tilde{v} \in S^{-1}v} \frac{f(\tilde{v})}{|S'(\tilde{v})|} e^{-s \log g(\tilde{v})} , \tag{2.1}$$
and let $\rho(\mathcal{L}_s)$ be its spectral radius. Define $\psi(s) = \log \rho(\mathcal{L}_s)$, and observe that $\psi(s)$ is the topological pressure of the potential $-\log |S'| - s \log g$ under the dynamics of S [14]. [2]

The operator $\mathcal{L}_0$ is the usual Perron-Frobenius operator of S, so $\psi(0) = 0$ and $\psi'(0) = -\int \log g \, d\mu_{\text{ac}} < 0$, see e.g. [14]. From the assumption in Hypothesis 3 that there is also a measure μ_- with $-\int \log g \, d\mu_- > 0$, it follows that $\psi(s) \to \infty$ as $s \to \infty$. Because of its convexity, $\psi(s)$ has therefore a unique further zero $s_* > 0$. This number characterizes the distribution of "small values" of φ_∞ in the sense of the following theorem.

Theorem 1.
$$\lim_{x \to \infty} \frac{1}{x} \log m\{v \in \mathbb{T}^1 : \log \varphi_\infty(v) < -x\} = -s_* . \tag{2.2}$$

[2] To be more precise, it is the pressure of the topological Markov chain that encodes S.

Replacing $-x$ by $\log\epsilon$, this can be reformulated as

$$\lim_{\epsilon\to 0}\frac{\log m\{\varphi_\infty<\epsilon\}}{\log\epsilon}=s_*\,. \tag{2.3}$$

For the local analysis of φ_∞ (see section 2.2) we also need a modification of this last identity. Define

$$\Xi_\epsilon:=\frac{1}{\epsilon}\int_{\mathbb{T}^1}\min\{\varphi_\infty(t),\epsilon\}\,dt\quad(\epsilon>0)\,, \tag{2.4}$$

so that

$$1-\Xi_\epsilon=\frac{1}{\epsilon}\int_{\mathbb{T}^1}(\epsilon-\varphi_\infty(t))^+\,dt\,. \tag{2.5}$$

Theorem 2.

$$\lim_{\epsilon\to 0}\frac{\log\Xi_\epsilon}{\log\epsilon}=0\quad\textit{and}\quad\lim_{\epsilon\to 0}\frac{\log(1-\Xi_\epsilon)}{\log\epsilon}=s_*\,. \tag{2.6}$$

The proofs of (slight generalisations of) these two theorems are provided in section 4.

2.2. **Local scaling properties.** As in [16] we define a local stability index $\sigma(v)$ of the invariant graph φ_∞ in the following way:

$$\sigma(v):=\sigma_+(v)-\sigma_-(v) \tag{2.7}$$

where

$$\sigma_-(v):=\lim_{\epsilon\to 0}\frac{\log\Sigma_\epsilon(v)}{\log\epsilon}\quad\text{and}\quad\sigma_+(v):=\lim_{\epsilon\to 0}\frac{\log(1-\Sigma_\epsilon(v))}{\log\epsilon} \tag{2.8}$$

with

$$\Sigma_\epsilon(v):=\frac{1}{\epsilon\cdot|U_\epsilon(v)|}\int_{U_\epsilon(v)}\min\{\varphi_\infty(t),\epsilon\}\,dt \tag{2.9}$$

and

$$1-\Sigma_\epsilon(v)=\frac{1}{\epsilon\cdot|U_\epsilon(v)|}\int_{U_\epsilon(v)}(\epsilon-\varphi_\infty(t))^+\,dt\,. \tag{2.10}$$

The $U_\epsilon(v):=(v-\epsilon,v+\epsilon)$ are symmetric interval neighbourhoods of v of size 2ϵ.

Of course, the limits in (2.8) need not exist *a priori*, but sufficient conditions for their existence are formulated in Theorem 3. If $\sigma_+(v)$ and $\sigma_-(v)$ both exist, they are non-negative and at most one of them can be strictly positive.

For $\theta\in\Theta$ we define $\hat\sigma_\pm(\theta)=\sigma_\pm(\Pi\theta)$.

Proposition 2. *$\hat\sigma_\pm(\hat S\theta)=\hat\sigma_\pm(\theta)$ for all $\theta\in\Theta$.*

This is essentially Theorem 2.2 of [16]. Observe just that the proof of that theorem applies to any forward and backward invariant set.

Corollary 1. *For each ergodic $\hat S$-invariant measure $\hat\mu$ the function $\hat\sigma_\pm$ is $\hat\mu$-a.s. constant.*

Recall from Hypotheis 2 that $\mathcal{U}_n(v)$ denotes the family of all interval neighbourhoods U of $v\in\mathbb{T}^1$ such that $S^n_{|U}:U\to S^nU$ is a diffeomorphism. The following theorem is proved in section 5.

Theorem 3. *Let $v \in \mathbb{T}^1$ be* regular *in the sense that*

$$\Gamma(v) := \lim_{n\to\infty} \frac{1}{n} \log g_n(v) \quad \textit{and} \quad \Lambda(v) := \lim_{n\to\infty} \frac{1}{n} \log |(S^n)'(v)| \tag{2.11}$$

exist and that

there are sequences $n_1 < n_2 < \dots$ of integers and $U_{\epsilon_1} \supseteq U_{\epsilon_2} \supseteq \dots$ of symmetric *interval neighbourhoods of v with $U_{\epsilon_k} \in \mathcal{U}_{n_k}(v)$ such that*

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = 1 \quad \textit{and} \quad \Delta := \inf_{k\geqslant 1} |S^{n_k} U_{\epsilon_k}| > 0\,. \tag{2.12}$$

1. If $\Gamma(v) + \Lambda(v) > 0$, then

$$\sigma_+(v) = \frac{\Gamma(v) + \Lambda(v)}{\Lambda(v)} \cdot s_* \quad \textit{and} \quad \sigma_-(v) = 0\,. \tag{2.13}$$

2. If $\Gamma(v) + \Lambda(v) < 0$, then

$$\sigma_-(v) = -\frac{\Gamma(v) + \Lambda(v)}{\Lambda(v)} \quad \textit{and} \quad \sigma_+(v) = 0\,. \tag{2.14}$$

Remark 5 (On the notion of regularity of a point v)**.**

a) The set of points $v \in \mathbb{T}^1$ for which (2.11) is violated has measure zero for each S-invariant measure by Birkhoff's Ergodic Theorem. Those points for for which (2.12) is violated have measure zero for each S-invariant Gibbs measure. Indeed, in section 6.2 we prove the stronger fact that the same is true for each S-invariant measure μ with the property that

$$\mu(W_\epsilon) = \mathcal{O}\left(\left(\log\log\frac{1}{\epsilon}\right)^{-(1+q)}\right) \quad \text{as } \epsilon \to 0 \text{ for some } q > 0 \tag{2.15}$$

where W_ϵ is the ϵ-neighbourhood of the set of endpoints of monotonicity intervals of S. (Observe that for each Gibbs measure μ there exists $t \in (0,1)$ such that $\mu(W_\epsilon) = \mathcal{O}(\epsilon^t)$, because S is piecewise uniformly expanding.)

b) If S is an expanding C^{1+}-map of $\mathbb{T}^1$, then there is, for each $n \geqslant 1$, a symmetric interval $U \in \mathcal{U}_n(v)$ with $|S^n U| = 1$. Therefore (2.12) is satisfied for all $v \in \mathbb{T}^1$ in this case.

c) If one replaces the symmetric intervals in the definition of $\Sigma_\epsilon(v)$ by maximal monotonicity intervals, then (2.12) is satisfied for all Markov maps.

Remark 6. Numerical investigations related to equations (2.13) and (2.14) are presented in [8].

Remark 7. In [9] we characterize the Hausdorff and packing dimension of the set $\{\theta \in \Theta : \hat{\varphi}_\infty(\theta) = 0\}$ and related ones using thermodynamic formalism for the map S. In other words, we study the local scaling behaviour of the set of zeros of $\hat{\varphi}_\infty$. Theorems 1 - 3 extend this point of view in that they describe the local scaling behaviour of the subgraph of $\hat{\varphi}_\infty$ in regions where $\hat{\varphi}_\infty$ assumes values very close to zero.

2.3. **The Anosov case.** In Example 2 we described how Anosov surface diffeomorphisms driving a Hölder function $\hat{g} : \mathbb{T}^2 \to (0,\infty)$ fit the general framework of this note. The basic observation is Proposition 1 relating the invariant graph $\hat{\varphi}_\infty$ defined in (1.3) to its "one-sided" approximation $\varphi_\infty \circ \Pi$ which is the invariant graph for the system where the multiplier function $\hat{g}$ is replaced by $g \circ \Pi$.

Using Proposition 1 and standard facts about Anosov surface diffeomorphisms, in particular that the stable and the unstable foliation are uniformly transversal and C^{1+} [13, Theorem III.3.1], one can deduce the following theorem from the results of the previous two subsections.

Recall from Example 2 that $\hat\mu^-_{\text{SRB}}$ is the Sinai-Ruelle-Bowen measure of $\hat S^{-1}$ and denote by $\hat\psi(t)$ the topological pressure of $-\log|D_u\hat S^{-1}| - t\log\hat g$ under $\hat S^{-1}$. As $\log\hat g$ is cohomologous to $\log g\circ\Pi$ by (1.13) and $\log|D_u\hat S^{-1}|$ to $\log|S'|\circ\Pi$ by (1.15), we have

$$\hat\psi(s) = \psi(s) \quad\text{and}\quad \hat\psi'(0) = -\hat\mu^-_{\text{SRB}}(\log\hat g) = -\mu_{\text{ac}}(\log g) < 0 \tag{2.16}$$

so that the zero $s_* > 0$ of ψ defined in section 2.1 is at the same time the unique positive zero of $\hat\psi$.

Theorem 4. *Let $\Theta = \mathbb{T}^2$ and let $\hat S : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^2 Anosov diffeomorphism. Suppose that $g : \mathbb{T}^2 \to (0,\infty)$ is Hölder continuous. Then the invariant graph $\hat\varphi_\infty$ has the following properties:*

1.

$$\lim_{\epsilon\to 0}\frac{\log m^2\{\hat\varphi_\infty < \epsilon\}}{\log\epsilon} = s_*\,. \tag{2.17}$$

2.

$$\lim_{\epsilon\to 0}\frac{\log\hat\Xi_\epsilon}{\log\epsilon} = 0 \quad\textit{and}\quad \lim_{\epsilon\to 0}\frac{\log(1-\hat\Xi_\epsilon)}{\log\epsilon} = s_* \tag{2.18}$$

where $\hat\Xi_\epsilon := \frac{1}{\epsilon}\int_{\mathbb{T}^2}\min\{\hat\varphi_\infty,\epsilon\}\,dm^2$ $(\epsilon > 0)$, so that $1-\hat\Xi_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{T}^2}(\epsilon-\hat\varphi_\infty)^+\,dm^2$.

Furthermore, there is a measurable subset $\Theta_0 \subseteq \Theta$, which has measure zero for each Gibbs measure of T, such that for each $\theta\in\Theta\setminus\Theta_0$ the limits

$$\hat\Gamma(\theta) := \lim_{n\to\infty}\frac{1}{n}\log\hat g_n(\theta) \quad\textit{and}\quad \hat\Lambda(\theta) := \lim_{n\to\infty}\frac{1}{n}\log|D_u\hat S^{-n}(\theta)| \tag{2.19}$$

exist and satisfy $\hat\Gamma(\theta) = \Gamma(\Pi\theta)$ and $\hat\Lambda(\theta) = \Lambda(\Pi\theta)$, and the following holds:

3. If $\hat\Gamma(\theta)+\hat\Lambda(\theta) > 0$, then

$$\lim_{\epsilon\to 0}\frac{\log(1-\Sigma_\epsilon(\theta))}{\log\epsilon} = \frac{\hat\Gamma(\theta)+\hat\Lambda(\theta)}{\hat\Lambda(\theta)}\cdot s_* \quad\textit{and}\quad \lim_{\epsilon\to 0}\frac{\log\Sigma_\epsilon(\theta)}{\log\epsilon} = 0 \tag{2.20}$$

where $\Sigma_\epsilon(\theta) := \frac{1}{\epsilon\cdot|U_\epsilon(\theta)|}\int_{U_\epsilon(\theta)}\min\{\hat\varphi_\infty,\epsilon\}\,dm^2$ and $U_\epsilon(\theta)$ is a ϵ-neighbourhood of θ in $\mathbb{T}^2$, so that $1-\Sigma_\epsilon(\theta) = \frac{1}{\epsilon\cdot|U_\epsilon(\theta)|}\int_{U_\epsilon(\theta)}(\epsilon-\hat\varphi_\infty)^+\,dm^2$.

4. If $\hat\Gamma(\theta)+\hat\Lambda(\theta) < 0$, then

$$\lim_{\epsilon\to 0}\frac{\log\Sigma_\epsilon(\theta)}{\log\epsilon} = -\frac{\hat\Gamma(\theta)+\hat\Lambda(\theta)}{\hat\Lambda(\theta)} \quad\textit{and}\quad \lim_{\epsilon\to 0}\frac{\log(1-\Sigma_\epsilon(\theta))}{\log\epsilon} = 0\,. \tag{2.21}$$

Proof. The existence of the limits in (2.19) is again a consequence of Birkhoff's theorem. The identities $\hat\Gamma(\theta) = \Gamma(\Pi\theta)$ and $\hat\Lambda(\theta) = \Lambda(\Pi\theta)$ follow from the fact that $\log\hat g$ is cohomologous to $\log g\circ\Pi$ and $\log|D_u\hat S^{-1}|$ to $\log|S'|\circ\Pi$, see the discussion before the theorem. In view of Remark 5a we can choose Θ_0 such that all points in $\Theta\setminus\Theta_0$ are regular in the sense of Theorem 3. Then all other claims follow from Theorems 1 - 3 along the following lines: Let $c = \frac{a}{h(a)}e^{2\|\hat b\|_\infty}$. Then

$$\Pi^{-1}\{\varphi_\infty < c^{-1}\epsilon\} \subseteq \{\hat\varphi_\infty < \epsilon\} \subseteq \Pi^{-1}\{\varphi_\infty < c\epsilon\} \tag{2.22}$$

and
$$c\,(c^{-1}\epsilon - \varphi_\infty\circ\Pi)^+ \leqslant (\epsilon-\hat\varphi_\infty)^+ \leqslant c^{-1}(c\epsilon-\varphi_\infty\circ\Pi)^+ \tag{2.23}$$
because of Proposition 1. Therefore it suffices to prove (2.17) and (2.18) for the graph $\varphi_\infty\circ\Pi$ instead of $\hat\varphi_\infty$. As $\varphi_\infty\circ\Pi$ is constant along local stable manifolds, and as the passage to local coordinates is absolutely continuous with bounded Jacobian determinant (see [3, Proposition 4.2] for details), there is a constant $C>0$ such that
$$C^{-1}\leqslant \frac{m^2\{\varphi_\infty\circ\Pi<\epsilon\}}{m\{\varphi_\infty<\epsilon\}}\ ,\ \frac{\int_{\mathbb{T}^2}(\epsilon-\varphi_\infty\circ\Pi)^+\,dm^2}{\int_{\mathbb{T}^1}(\epsilon-\varphi_\infty)^+\,dm}\leqslant C\,. \tag{2.24}$$
Now (2.17) and (2.18) follow from Theorem 1 and 2, respectively. With essentially the same arguments, (2.20) and (2.21) both follow from Theorem 3. □

3. Distortion estimates

3.1. **Branch distortion.** Recall that $\mathcal{U}_n(v)$ denotes the family of all interval neighbourhoods U of $v\in\mathbb{T}^1$ such that $S^n_{|U}:U\to S^nU$ is a diffeomorphism.

The following proposition is most important for estimating distortions along single branches $F^n_\theta:I\to I$. It uses only the concavity of $h:I\to I$. As
$$0<c_h:=\min\{a,h'(a)\}\leqslant\min\{h'(x):x\in I\}\,, \tag{3.1}$$
there is a constant $a_h>0$ such that
$$h'(x)\geqslant e^{-a_hx}\text{ for all }x\in I\,. \tag{3.2}$$
We will also use the following notation: For $n\geqslant 1$ and $v\in\mathbb{T}^1$ define $f_{n,v}:\mathbb{T}^1\to\mathbb{T}^1$ by $f_{1,v}(x)=g(Sv)h(x)$ and $f_{n,v}(x)=f_{1,v}(f_{n-1,Sv}(x))$ if $n>1$. Observe that $f_{n,v}(x)=f_{n-1,v}(f_{1,S^{n-1}v}(x))$. By definition, $f_{n,v}(x)$ is always to be interpreted as a point in the fibre over v. Observe also that $F^n_{\hat S^{-n}\theta}(x)=f_{n,\Pi\theta}(x)$ for all $n\geqslant1$ and $\theta\in\Theta$.

For fixed $n\in\mathbb{N}$ and $x\in I$ let
$$x_{-i}=f_{n-i,S^iv}(x)\ (i=1,\dots,n)\,, \tag{3.3}$$
and observe that $x_{-n}=x$ is a point in the fibre over S^nv, i.e. at time $-n$, while x_0 is a point in the fibre over v, i.e. at time 0. Note that we suppress the n-dependence of x_i in this notation.

For a given sequence $(\alpha_i)_{i\geqslant1}$ of positive real numbers let $A_n=\sum_{i=1}^n\alpha_i$ and set $C_n=c_h^{-1}\,e^{a_hA_n}$. If the sequence is summable we extend this notation to $A_\infty=\sum_{i=1}^\infty\alpha_i$ and $C_\infty=c_h^{-1}\,e^{a_hA_\infty}$.

Proposition 3. *Let $(\alpha_i)_{i\geqslant1}$ be a sequence of positive real numbers. For all $n\in\mathbb{N}$, $v\in\mathbb{T}^1$ and $x\in I$,*
$$\exp\left(-a_h\sum_{i=1}^n x_{-i}\right)\leqslant\frac{f'_{n,v}(x)}{f'_{n,v}(0)}\leqslant 1\,, \tag{3.4}$$
and if
$$x_0=f_{n,v}(x)\leqslant C_n^{-1}\,\alpha_i\,g_i(v)\ (i=1,\dots,n)\,, \tag{3.5}$$
then
$$\sum_{i=1}^n x_{-i}\leqslant C_n\,x_0\sum_{i=1}^n g_i(v)^{-1}\leqslant A_n\,. \tag{3.6}$$

Proof. The second inequality of (3.4) is an immediate consequence of the concavity of the branches. The first one follows from

$$\begin{aligned} f'_{n,v}(x) &= f'_{1,v}(x_{-1}) \cdot f'_{n-1,Sv}(x_{-n}) = \ \dots \ = \prod_{i=0}^{n-1} f'_{1,S^i v}(x_{-i-1}) \\ &= \prod_{i=1}^{n} g(S^i v) \cdot \prod_{i=1}^{n} h'(x_{-i}) \geqslant f'_{n,v}(0) \cdot \exp\left(-a_h \sum_{i=1}^{n} x_{-i}\right) . \end{aligned} \tag{3.7}$$

In order to prove (3.6), it suffices to show that

$$x_{-i} \leqslant C_n \, x_0 \, g_i(v)^{-1} \leqslant \alpha_i \ (i = 1, \dots, n) . \tag{3.8}$$

As the second inequality is just a reformulation of (3.5), it remains to prove the first one.

For $i = 1, \dots, n$ we have

$$x_0 = f_{i,v}(x_{-i}) \geqslant x_{-i} \cdot f'_{i,v}(x_{-i})$$

and, as in (3.7),

$$f'_{i,v}(x_{-i}) = \prod_{j=1}^{i} g(S^j v) \cdot \prod_{j=1}^{i} h'(x_{-j}) \geqslant g_i(v) \, h'(x_{-i}) \cdot \exp\left(-a_h \sum_{j=1}^{i-1} x_{-j}\right) . \tag{3.9}$$

Hence

$$x_{-i} \, h'(x_{-i}) \leqslant x_0 \, g_i(v)^{-1} \cdot \exp\left(a_h \sum_{j=1}^{i-1} x_{-j}\right) , \tag{3.10}$$

and as $x_0 \leqslant c_h e^{-a_h A_n} \alpha_i g_i(v)$ for $i = 1, \dots, n$ by assumption (3.5), it follows that

$$x_{-i} \leqslant \alpha_i \cdot \exp\left(-a_h A_n + a_h \sum_{j=1}^{i-1} x_{-j}\right) . \tag{3.11}$$

For $i = 1$ we see at once that $x_{-1} \leqslant \alpha_1 \, e^{-a_h A_n} \leqslant \alpha_1$, and for $i = 2, \dots, n$ it follows inductively that

$$x_{-i} \leqslant \alpha_i \cdot \exp\left(-a_h A_n + a_h \sum_{j=1}^{i-1} \alpha_j\right) \leqslant \alpha_i . \tag{3.12}$$

Combined with (3.10) this yields (3.8), namely

$$x_{-i} \leqslant x_0 \, g_i(v)^{-1} \, c_h^{-1} \, e^{a_h A_n} = C_n \, x_0 \, g_i(v)^{-1} . \tag{3.13}$$

□

Corollary 2. *Let $(\alpha_i)_{i \geqslant 1}$ be as in the preceding proposition and suppose that $\alpha_i \leqslant 1$ for all i. Then, for all $n \in \mathbb{N}$ and $v \in \mathbb{T}^1$, there exists $i \in \{1, \dots, n\}$ such that*

$$\varphi_n(v) > C_\infty^{-1} \, \alpha_i \, g_i(v) . \tag{3.14}$$

Proof. Suppose for a contradiction that there are $n \in \mathbb{N}$ and $v \in \mathbb{T}^1$ such that

$$\varphi_n(v) \leqslant C_\infty^{-1} \, \alpha_i \, g_i(v) \quad (i = 1, \dots, n) . \tag{3.15}$$

Now Proposition 3 implies

$$\varphi_n(v) = f_{n,v}(a) \geqslant a \, f'_{n,v}(a) \geqslant a \, g_n(v) \, e^{-a_h A_n} \geqslant c_h \, g_n(v) \, e^{-a_h A_n} \, \alpha_n = C_n^{-1} \, g_n(v) \, \alpha_n \tag{3.16}$$

which contradicts (3.15) for $i = n$, because $C_n < C_\infty$. $\square$

3.2. **Area distortion.** Here are some consequences of the estimates from the previous section for "telescoping" certain small areas in $M = \mathbb{T}^1 \times I$. Recall that D is the distortion constant from Hypothesis 2 and Remark 2. Denote also by m^2 the 2-dimensional Lebesgue measure on $\mathbb{T}^1 \times I$. For $n \in \mathbb{N}$ and $U \in \mathcal{U}_n(v)$ we define the maps

$$f_{n,U} : S^n(U) \times I \to M, \quad (v, x) \mapsto ((S^n{}_{|U})^{-1}(v), f_{n,v}(x)) . \tag{3.17}$$

Proposition 4. *In the situation of Proposition 3, let $(\alpha_i)_{i\geqslant 1}$ be a* summable *sequence. Then for all $v \in \mathbb{T}^1$, all $n \in \mathbb{N}$, all $U \in \mathcal{U}_n(v)$, all $H > 0$ such that*

$$\left|(S^n)'(v) \cdot g_i(v)\right|^{-1} \leqslant H^{-1} D^{-1} C_\infty^{-1} \alpha_i \text{ for } (i = 1, \dots, n) , \tag{3.18}$$

and for $\tilde{v} \in U$ and $\tilde{x} \in I$ with

$$f_{n,\tilde{v}}(\tilde{x}) \leqslant H \cdot |(S^n)'(v)|^{-1} \tag{3.19}$$

the following holds:
1.

$$e^{-a_h A_\infty} \leqslant \frac{f'_{n,\tilde{v}}(\tilde{x})}{f'_{n,\tilde{v}}(0)} \leqslant 1 . \tag{3.20}$$

2.

$$D^{-1} e^{-a_h A_\infty} \leqslant \frac{f'_{n,\tilde{v}}(\tilde{x})}{f'_{n,v}(0)} \leqslant D . \tag{3.21}$$

3. For the Jacobian $Jf_{n,U}$,

$$D^{-2} e^{-a_h A_\infty} \leqslant \frac{Jf_{n,U}(S^n\tilde{v}, \tilde{x})}{Jf_{n,U}(S^n v, 0)} \leqslant D^2 . \tag{3.22}$$

4. For measurable $V, W \subseteq S^n(U) \times I$,

$$\left(D^4 e^{a_h A_\infty}\right)^{-1} \leqslant \frac{m^2(V)}{m^2(W)} \Big/ \frac{m^2(f_{n,U} V)}{m^2(f_{n,u} W)} \leqslant D^4 e^{a_h A_\infty} . \tag{3.23}$$

Proof. 1. This follows from Proposition 3 once we have checked that $f_{n,\tilde{v}}(\tilde{x}) \leqslant C_\infty^{-1} \alpha_i g_i(\tilde{v})$ for $i = 1, \dots, n$: By (3.19), (3.18) and Hypothesis 2,

$$f_{n,\tilde{v}}(\tilde{x}) \leqslant H \cdot |(S^n)'(v)|^{-1} \leqslant D^{-1} g_i(v) C_\infty^{-1} \alpha_i \leqslant g_i(\tilde{v}) C_\infty^{-1} \alpha_i . \tag{3.24}$$

2. As

$$\frac{f'_{n,\tilde{v}}(\tilde{x})}{f'_{n,v}(0)} = \frac{f'_{n,\tilde{v}}(\tilde{x})}{f'_{n,\tilde{v}}(0)} \cdot \frac{g_n(\tilde{v})}{g_n(v)} ,$$

this follows at once from Hypothesis 2 and (3.20).

3. Due to the skew product structure of $f_{n,U}$, its Jacobian is

$$Jf_{n,U}(S^n v, x) = |(S^n)'(v)|^{-1} f'_{n,v}(x) . \tag{3.25}$$

Hence

$$\frac{Jf_{n,U}(S^n\tilde{v}, \tilde{x})}{Jf_{n,U}(S^n v, 0)} = \frac{|(S^n)'(v)|}{|(S^n)'(\tilde{v})|} \cdot \frac{f'_{n,\tilde{v}}(\tilde{x})}{f'_{n,v}(0)} ,$$

and (3.22) follows at once from Remark 2 and (3.21).

4. This is an immediate consequence of (3.22). $\square$

4. The distribution of φ_∞: Proofs

4.1. **Proof of Theorem 1.** The proof of Theorem 1 is inspired by proofs of a related result in queuing theory, namely the determination of Loyne's exponent [12] for the stationary distribution of Lindley's recursion [11], see also [5] and in particular [10, Lemmas 4 and 5].

Recall the weighted Perron-Frobenius operators $\mathcal{L}_s$ defined in (2.1) and the notation $\psi(s) = \log\rho(\mathcal{L}_s)$. We noticed already that $\psi(0)=0$, $\psi'(0)<0$, and that there is a unique $s_*>0$ such that $\psi(s_*)=0$ and $\psi'(s_*)>0$. For technical reasons we prove a slightly stronger statement than Theorem 1, namely: For each family $(J_x)_{x>0}$ of subintervals of $\mathbb{T}^1$ with $\inf_{x>0}|J_x|>0$ we have

$$\lim_{x\to\infty}\frac{1}{x}\log m\{v\in J_x:\log\varphi_\infty(v)<-x\}=-s_*\,. \tag{4.1}$$

Fix any $s\in(0,s_*)$ and choose $\delta>0$ such that $\rho(\mathcal{L}_s)e^{3s\delta}<1$. There is a constant $C>0$ that depends on s and δ such that

$$\|\mathcal{L}_s^n 1\|_1\leqslant C\left(\rho(\mathcal{L}_s)e^{s\delta}\right)^n\leqslant Ce^{-2ns\delta}\quad\text{for all } n\geqslant 1\,. \tag{4.2}$$

For $\kappa>0$ denote

$$A_\kappa=\left\{v\in\mathbb{T}^1:\exists n\geqslant 1\text{ such that } g_n(v)\leqslant\kappa e^{n\delta}\right\}\,. \tag{4.3}$$

Lemma 1. *There is a constant $C>0$ that depends on t and δ such that for all $\kappa>0$*

$$m(A_\kappa)\leqslant C\cdot\kappa^s\,. \tag{4.4}$$

Proof. As $s>0$, we have the usual Cramér type estimate for each $n\geqslant 1$:

$$\begin{aligned}
m\left\{v\in\mathbb{T}^1:g_n(v)\leqslant\kappa e^{n\delta}\right\}&=m\left\{v\in\mathbb{T}^1:\kappa^s e^{ns\delta}\,e^{-s\log g_n(v)}\geqslant 1\right\}\\
&\leqslant\kappa^s e^{ns\delta}\int_{\mathbb{T}^1}e^{-s\log g_n}\,dm\\
&=\kappa^s e^{ns\delta}\int_{\mathbb{T}^1}\mathcal{L}_0^n(e^{-s\log g_n})\,dm=\kappa^s e^{ns\delta}\int_{\mathbb{T}^1}\mathcal{L}_s^n(1)\,dm\\
&\leqslant Ce^{-ns\delta}\cdot\kappa^s\,.
\end{aligned}\tag{4.5}$$

Summing this inequality over all $n=1,2,\dots$, we get (4.4) with the constant $C/(e^{s\delta}-1)$, which depends again only on δ and s. □

We start the proof of (4.1) with the upper estimate. Let $\alpha_i=e^{-i\delta}$ $(i=1,2,\dots)$ so that $A_\infty=\sum_{i=1}^\infty\alpha_i=\frac{1}{e^\delta-1}$ and $C_\infty=c_h^{-1}e^{-a_hA_\infty}$ depend only on δ. Let $v\in\mathbb{T}^1\setminus A_\kappa$. Then $g_i(v)\alpha_i>\kappa$ for all $i\geqslant 1$. Therefore, by Corollary 2, for all $n\in\mathbb{N}$ there exists $i\in\{1,\dots,n\}$ such that

$$\varphi_n(v)>C_\infty^{-1}\,\alpha_i\,g_i(v)>C_\infty^{-1}\,\kappa \tag{4.6}$$

and hence

$$\varphi_\infty(v)=\inf_{n\geqslant 1}\varphi_n(v)\geqslant C_\infty^{-1}\,\kappa\,. \tag{4.7}$$

Now fix $x>0$ and let $\kappa=e^{-x}C_\infty$. Then $\varphi_\infty(v)\geqslant e^{-x}$ for $v\in\mathbb{T}^1\setminus A_\kappa$ so that, in view of Lemma 1,

$$\limsup_{x\to\infty}\frac{1}{x}\log m\{v\in J_x:\log\varphi_\infty(v)<-x\}\leqslant\limsup_{x\to\infty}\frac{1}{x}\log m(A_{e^{-x}C_\infty})=-s\,. \tag{4.8}$$

As this estimate applies to each $s\in(0,s_*)$, this proves the upper estimate in (4.1).

We turn to the lower estimate. As $\varphi_\infty(v) \leqslant \varphi_n(v) = f_{n,v}(a) \leqslant a\, g_n(v)$ for all n and all $v \in \mathbb{T}^1$, we have immediately that

$$m\{v \in J_x : \log\varphi_\infty(v) < -x\} \geqslant m\{v \in J_x : \log g_n(v) < -x - \log a\} \tag{4.9}$$

for all $n \geqslant 1$. Let $\alpha := \psi'(s_*) > 0$. Then, for $n = \lceil \alpha^{-1}(x + \log a)\rceil$,

$$\begin{aligned}\liminf_{x\to\infty} \frac{1}{x} \log m\{v \in J_x : \log\varphi_\infty(v) < -x\} &\geqslant \liminf_{n\to\infty} \frac{1}{\alpha n} \log \frac{m\{v \in J_x : -\log g_n(v) > n\alpha\}}{m(J_x)} \\ &= \frac{1}{\alpha}(\psi(s_*) - s_*\alpha) = \frac{\psi(s_*)}{\psi'(s_*)} - s_* = -s_* \,. \end{aligned} \tag{4.10}$$

This is a consequence of large deviations theory for the map S, details of which are provided in the appendix. Together with the upper estimate (4.8), it finishes the proof of Theorem 1.

4.2. **Proof of Theorem 2.** Again we prove a "localized" version of this theorem: instead of the quantity $\Xi_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{T}^1} \min\{\varphi_\infty(t), \epsilon\}\, dt$ we look at

$$\Xi_\epsilon := \frac{1}{\epsilon}\int_{J_\epsilon} \min\{\varphi_\infty(t), \epsilon\}\, dt \tag{4.11}$$

for a family of intervals J_ϵ with $\inf_\epsilon |J_\epsilon| > 0$.

We only have to show that

$$\lim_{\epsilon\to 0} \frac{\log(1-\Xi_\epsilon)}{\log\epsilon} = s_* > 0\,, \tag{4.12}$$

because this implies at once that $\lim_{\epsilon\to 0} \frac{\log\Xi_\epsilon}{\log\epsilon} = 0$. Recall that

$$1 - \Xi_\epsilon = \frac{1}{\epsilon}\int_{J_\epsilon} (\epsilon - \varphi_\infty(v))^+\, dv \leqslant m\{v \in J_\epsilon : \varphi_\infty(v) \leqslant \epsilon\}\,. \tag{4.13}$$

Therefore we conclude from (4.1) that

$$\limsup_{\epsilon\to 0} \frac{\log(1-\Xi_\epsilon)}{\log\epsilon} \leqslant \limsup_{\epsilon\to 0} \frac{1}{\log\epsilon} \log m\{v \in J_\epsilon : \varphi_\infty(v) \leqslant \epsilon\} = s_*\,. \tag{4.14}$$

For the lower estimate observe that

$$1 - \Xi_\epsilon = \frac{1}{\epsilon}\int_{J_\epsilon} (\epsilon - \varphi_\infty(v))^+\, dv \geqslant \frac{1}{2} m\left\{v \in J_\epsilon : \varphi_\infty(v) \leqslant \epsilon/2\right\}\,. \tag{4.15}$$

This implies, by (4.1) again,

$$\liminf_{\epsilon\to 0} \frac{\log(1-\Xi_\epsilon)}{\log\epsilon} \geqslant \liminf_{\epsilon\to 0} \frac{1}{\log(\epsilon/2)} \log m\{v \in J_\epsilon : \varphi_\infty(v) \leqslant \epsilon/2\} = s_*\,. \tag{4.16}$$

5. The stability index: Proof of Theorem 3

Let U_{ϵ_k} be a symmetric open interval neighbourhood of v in $\mathcal{U}_{n_k}(v)$ satisfying the regularity assumption (2.12). As $1 \geqslant \int_{U_{\epsilon_k}} |(S^{n_k})'|\, dm = |S^{n_k} U_{\epsilon_k}| \geqslant \Delta$, it follows from Remark 2 that

$$\frac{\Delta}{2D} \leqslant \epsilon_k |(S^{n_k})'(v)| \leqslant \frac{D}{2} = H\,. \tag{5.1}$$

Combining this with (2.11) and (2.12) we obtain

$$\lim_{k\to\infty} \frac{\log\epsilon_{k+1}}{\log\epsilon_k} = \lim_{k\to\infty} \frac{\log|(S^{n_{k+1}})'(v)|}{\log|(S^{n_k})'(v)|} = \lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1\,. \tag{5.2}$$

For each $\epsilon \in [\epsilon_{k+1}, \epsilon_k]$ we have

$$\frac{\log \epsilon_{k+1}}{\log \epsilon_k} \frac{\log \Sigma_{\epsilon_{k+1}}(v)}{\log \epsilon_{k+1}} \leqslant \frac{\log \Sigma_\epsilon(v)}{\log \epsilon} \leqslant \frac{\log \epsilon_k}{\log \epsilon_{k+1}} \frac{\log \Sigma_{\epsilon_k}(v)}{\log \epsilon_k} \ , \tag{5.3}$$

and the same holds when $\Sigma_\epsilon(v)$ is replaced by $(1 - \Sigma_\epsilon(v))$. Therefore it suffices to evaluate the limits for $\sigma_\pm(v)$ in (2.8) only along the sequence $(\epsilon_k)_{k\in\mathbb{N}}$.

1. *The case* $\Gamma(v) + \Lambda(v) > 0$*:* We check the assumptions of Proposition 4: Let

$$\delta = \frac{1}{4} \min \{\Lambda(v), \Gamma(v) + \Lambda(v)\} \tag{5.4}$$

and observe that $\delta > 0$. As v is regular, there is a constant $C_v > 0$ such that $g_k(v) > C_v\, e^{k(\Gamma(v)-\delta)}$ and $|(S^k)'(v)| > C_v\, e^{k(\Lambda(v)-\delta)}$ for all $k \in \mathbb{N}$. Set $\alpha_i = e^{-i\delta}$. Then

$$\begin{aligned} |(S^n)'(v) \cdot g_i(v)|^{-1} &\leqslant C_v^{-2} e^{-n(\Lambda(v)-\delta)-i(\Gamma(v)-\delta)} \\ &\leqslant C_v^{-2} e^{-(n-i)3\delta - i2\delta} = C_v^{-2} e^{-n\delta} \alpha_i e^{-2(n-i)\delta} \\ &\leqslant C_v^{-2} e^{-n\delta} \alpha_i \end{aligned} \tag{5.5}$$

for all $n \in \mathbb{N}$ and all $i = 1, \dots, n$.

Now fix the constant H from Proposition 4 as $H = \frac{D}{2}$, where D is the basic distortion constant from Hypothesis 2 and Remark 2. Then, for all sufficiently large n, assumption (3.18) is satisfied for all $i = 1, \dots, n$. In particular,

$$|(S^n)'(v)|^{-1} \leqslant C_v^{-2} e^{-2n\delta} g_n(v)\,. \tag{5.6}$$

Assume for a contradiction that $f_{n_k,\tilde{v}}(a) \leqslant \epsilon_k$. Then $f_{n_k,\tilde{v}}(a) \leqslant H|(S^{n_k})'(v)|^{-1}$ by (5.1), so that also (3.19) is satisfied, and (3.21) of Proposition 4 yields

$$f_{n_k,\tilde{v}}(a) \geqslant a\, f'_{n_k,\tilde{v}}(a) \geqslant aD^{-1} e^{-a_h A_\infty} g_{n_k}(v) \geqslant aD^{-1} e^{-a_h A_\infty} C_v^2 e^{2n_k\delta} |(S^{n_k})'(v)|^{-1} \tag{5.7}$$

which contradicts $f_{n_k,\tilde{v}}(a) \leqslant H|(S^{n_k})'(v)|^{-1}$ when n_k is sufficiently large, say $n_k \geqslant N_0(v)$.

Therefore, $f_{n_k,\tilde{v}}(a) > \epsilon_k$ for all $n_k \geqslant N_0(v)$ and all $\tilde{v} \in U_{\epsilon_k}$, and there are functions $\delta_{n_k} : S^{n_k} U_{\epsilon_k} \to I$ such that $f_{n_k,\tilde{v}}(\delta_{n_k}(t)) = \epsilon_k \leqslant H\, |(S^{n_k})'(v)|^{-1}$ for all $t \in S^{n_k} U_{\epsilon_k}$. As $\frac{\epsilon_k}{\delta_{n_k}(t)} = \frac{f_{n_k,\tilde{v}}(\delta_{n_k}(t))}{\delta_{n_k}(t)} = f'_{n_k,\tilde{v}}(\tilde{x})$ for some $\tilde{x} = \tilde{x}(t)$, we conclude from (3.21) that

$$D^{-1} e^{-a_h A_\infty} \leqslant \frac{\epsilon_k}{\delta_{n_k}(t) g_{n_k}(v)} \leqslant D \quad \text{for all } t \in S^{n_k} U_{\epsilon_k}\,. \tag{5.8}$$

In view of (3.23) we have

$$\left(D^4 e^{a_h A_\infty}\right)^{-1} \leqslant \frac{\int_{S^{n_k} U_{\epsilon_k}} (\delta_{n_k}(t) - \varphi_\infty(t))^+ \, dt}{\int_{S^{n_k} U_{\epsilon_k}} \delta_{n_k}(t)\, dt} \Big/ \frac{\int_{U_{\epsilon_k}} (\epsilon_k - \varphi_\infty(t))^+ \, dt}{2\epsilon_k^2} \leqslant D^4 e^{a_h A_\infty}\,. \tag{5.9}$$

As the second quotient is just $1 - \Sigma_{\epsilon_k}(v)$, this implies

$$\sigma_+(v) = \lim_{k\to\infty} \frac{\log(1 - \Sigma_{\epsilon_k}(v))}{\log \epsilon_k} = \lim_{k\to\infty} \frac{1}{\log \epsilon_k} \cdot \log \frac{\int_{S^{n_k} U_{\epsilon_k}} (\delta_{n_k}(t) - \varphi_\infty(t))^+ \, dt}{\int_{S^{n_k} U_{\epsilon_k}} \delta_{n_k}(t)\, dt} \tag{5.10}$$

provided the last limit exists. Now let

$$\underline{\kappa}_k = D^{-1} \frac{\epsilon_k}{g_{n_k}(v)}, \quad \overline{\kappa}_k = De^{a_h A_\infty} \frac{\epsilon_k}{g_{n_k}(v)}$$

and observe that

$$\underline{\kappa}_k \leqslant \delta_{n_k}(t) \leqslant \overline{\kappa}_k \quad \text{for all } t \in S^{n_k} U_{\epsilon_k} \tag{5.11}$$

in view of (5.8). Therefore,

$$
\begin{aligned}
D^{-2}e^{-a_h A_\infty}\left(1-\Xi_{\underline{\kappa}_k}\right) &= \frac{\underline{\kappa}_k}{\overline{\kappa}_k}\left(1-\Xi_{\underline{\kappa}_k}\right) \\
&= \frac{1}{\overline{\kappa}_k}\int_{S^{n_k}U_{\epsilon_k}}\left(\underline{\kappa}_k-\varphi_\infty(t)\right)^+\,dt \\
&\leqslant \frac{\int_{S^{n_k}U_{\epsilon_k}}(\delta_{n_k}(t)-\varphi_\infty(t))^+\,dt}{\int_{S^{n_k}U_{\epsilon_k}}\delta_{n_k}(t)\,dt} \\
&\leqslant \Delta^{-1}\frac{1}{\underline{\kappa}_k}\int_{S^{n_k}U_{\epsilon_k}}\left(\overline{\kappa}_k-\varphi_\infty(t)\right)^+\,dt \\
&= \Delta^{-1}\frac{\overline{\kappa}_k}{\underline{\kappa}_k}\left(1-\Xi_{\overline{\kappa}_k}\right) \\
&= \Delta^{-1}D^2e^{a_h A_\infty}\left(1-\Xi_{\overline{\kappa}_k}\right).
\end{aligned}
\tag{5.12}
$$

As, in view of (2.11) and (5.1),

$$
\lim_{k\to\infty}\frac{\log\underline{\kappa}_k}{\log\epsilon_k}=\lim_{k\to\infty}\frac{\log\overline{\kappa}_k}{\log\epsilon_k}=1-\lim_{k\to\infty}\frac{\log g_{n_k}(v)}{\log\epsilon_k}=1+\frac{\Gamma(v)}{\Lambda(v)}>0\,,
\tag{5.13}
$$

and, observing (4.12),

$$
\lim_{k\to\infty}\frac{\log(1-\Xi_{\underline{\kappa}_k})}{\log\underline{\kappa}_k}=s_*=\lim_{k\to\infty}\frac{\log(1-\Xi_{\overline{\kappa}_k})}{\log\overline{\kappa}_k}\,,
\tag{5.14}
$$

we conclude from (5.10) and (5.12) that

$$
\sigma_+(v)=\lim_{k\to\infty}\left(\frac{\log(1-\Xi_{\underline{\kappa}_k})}{\log\underline{\kappa}_k}\cdot\frac{\log\underline{\kappa}_k}{\log\epsilon_k}\right)=s_*\cdot\frac{\Lambda(v)+\Gamma(v)}{\Lambda(v)}>0\,.
$$

In particular, $\sigma_-(v)=0$.

2. *The case* $\Gamma(v)+\Lambda(v)<0$*:* In this case,

$$
\lim_{k\to\infty}\frac{1}{n_k}\log\frac{g_{n_k}(v)}{\epsilon_{n_k}}=\Gamma(v)+\Lambda(v)<0\,.
\tag{5.15}
$$

As the branches $f_{n_k,\tilde v}$ are concave, it follows at once that

$$
\varphi_\infty(\tilde v)\leqslant\varphi_{n_k}(\tilde v)=f_{n_k,\tilde v}(a)\leqslant a g_{n_k}(\tilde v)<\epsilon_k e^{n_k(\Gamma(v)+\Lambda(v))/2}
\tag{5.16}
$$

uniformly for $\tilde v\in U_{\epsilon_k}$ when n_k is sufficiently large. This implies immediately that $\sigma_+(v)=0$.

In order to estimate of $\sigma_-(v)$ we will apply Proposition 3 directly. To this end we show that

$$
\sum_{i=1}^{n} f_{n-i,S^i\tilde v}(a)=o(n)\,.
\tag{5.17}
$$

Observe first that

$$
f_{n-i,S^i\tilde v}(a)\leqslant a\,g_{n-i}(S^i\tilde v)=a\frac{g_n(\tilde v)}{g_i(\tilde v)}\leqslant aD^2\frac{g_n(v)}{g_i(v)}\,.
\tag{5.18}
$$

Let

$$
\Delta(n):=\sup\left\{\frac{1}{i}|\log g_i-i\Gamma(v)|:i\geqslant n\right\}\,.
\tag{5.19}
$$

Then $\Delta(n) \to 0$ as $n \to \infty$. Fix a monotone sequence $(j_n)_n$ of integers with $j_n \to \infty$ and $j_n/n \to 0$, and define a second sequence $(\ell_n)_n$ as $\ell_n = \lfloor n\sqrt{\Delta(j_n)}\rfloor$. Then

$$\begin{aligned}\sum_{i=1}^{n} f_{n-i,S^i\tilde v}(a) &= \sum_{i=1}^{j_n} f_{n-i,S^i\tilde v}(a) + \sum_{i=j_n+1}^{n-\ell_n} f_{n-i,S^i\tilde v}(a) + \sum_{i=n-\ell_n+1}^{n} f_{n-i,S^i\tilde v}(a)\\ &\leqslant (j_n+\ell_n)a + aD^2 g_n(v)\sum_{i=j_n+1}^{n-\ell_n} g_i(v)^{-1}\\ &\leqslant (j_n+\ell_n)a + aD^2 e^{n(\Gamma(v)+\Delta(j_n))}\sum_{i=j_n+1}^{n-\ell_n} e^{i(-\Gamma(v)+\Delta(j_n))}\\ &\leqslant (j_n+\ell_n)a + aD^2\frac{e^{-\Gamma(v)+\Delta(j_n)}}{e^{-\Gamma(v)+\Delta(j_n)}-1}e^{n(\Gamma(v)+\Delta(j_n))+(n-\ell_n)(-\Gamma(v)+\Delta(j_n))}\\ &= o(n) + O(e^{2n\Delta(j_n)+\ell_n(\Gamma(v)-\Delta(j_n))})\,. \end{aligned} \tag{5.20}$$

As $\Gamma(v) < -\Lambda(v) < 0$ and as $n\Delta(j_n) = o(\ell_n)$, the $O(.)$-expression is bounded in n. So (5.17) is proved.

Now (3.4) of Proposition 3 shows that

$$e^{o(n_k)} \leqslant \frac{f'_{n_k,\tilde v}(x)}{g_{n_k}(\tilde v)} \leqslant 1 \tag{5.21}$$

uniformly for all $\tilde v \in U_{\epsilon_k}$ and all $x \in [0,a]$. In particular,

$$e^{o(n_k)} \leqslant \frac{f_{n_k,\tilde v}(a)}{g_{n_k}(\tilde v)} \leqslant a \tag{5.22}$$

uniformly for all $\tilde v \in U_{\epsilon_k}$.

We turn to the determination of $\sigma_-(v)$. As in the proof of Proposition 4 the distortion bound (5.21) implies analogous subexponential distortion bounds on the Jacobians $Jf_{n,U}$. Therefore, observing that $|S^{n_k}U_{\epsilon_k}| \geqslant \Delta > 0$,

$$e^{o(n_k)} = e^{o(n_k)}\frac{\int_{S^{n_k}U_{\epsilon_k}}\varphi_\infty(t)\,dt}{\int_{S^{n_k}U_{\epsilon_k}}a\,dt} \leqslant \frac{\int_{U_{\epsilon_k}}\varphi_\infty(\tilde v)\,d\tilde v}{\int_{U_{\epsilon_k}}\varphi_{n_k}(\tilde v)\,d\tilde v} \leqslant 1\,. \tag{5.23}$$

As $\log\epsilon_k = -n_k\Lambda(v) + o(n_k)$ and $\varphi_{n_k}(\tilde v) = f_{n_k,\tilde v}(a) = g_{n_k}(\tilde v)e^{o(n_k)}$ by (5.22), and as $\varphi_\infty(v) < \epsilon_k$ in view of (5.16), it follows from (5.23) that

$$\begin{aligned}\sigma_-(v) &= \lim_{k\to\infty}\frac{\log\Sigma_{\epsilon_k}(v)}{\log\epsilon_k} = \lim_{k\to\infty}\frac{1}{\log\epsilon_k}\left(\log\frac{\int_{U_{\epsilon_k}}\varphi_\infty(\tilde v)\,d\tilde v}{\epsilon_k|U_{\epsilon_k}|}\right)\\ &= \lim_{k\to\infty}\frac{1}{\log\epsilon_k}\log\frac{\int_{U_{\epsilon_k}}g_{n_k}(\tilde v)\,d\tilde v}{2\epsilon_k^2} = \lim_{k\to\infty}\frac{\log\epsilon_k^{-1}g_{n_k}(v)}{\log\epsilon_k}\\ &= -1 + \frac{\Gamma(v)}{-\Lambda(v)} = -\frac{\Gamma(v)+\Lambda(v)}{\Lambda(v)} > 0\,. \end{aligned} \tag{5.24}$$

6. Proofs for hyperbolic systems

6.1. **Proof of Proposition 1.** In the course of this proof we need the function $H(x) := \log\frac{h(x)}{x}$ which is well defined for $x \in (0,a]$ and which extends by continuity to $H(0) := 0$. Note also that $H(a) \leqslant H(x) < 0$ and $H'(x) = \frac{1}{h(x)}\left(h'(x) - \frac{h(x)}{x}\right) < 0$ for $x \in (0,a]$.

From the definition of F it follows that

$$\log F_{\hat{S}^{-1}\theta}(x) = \log x + \log \hat{g}(\hat{S}^{-1}\theta) + H(x) \tag{6.1}$$

for $x \in (0,a]$ and, by induction,

$$\log F^{\ell}_{\hat{S}^{-\ell}\theta}(x) = \log x + \sum_{k=1}^{\ell} \log \hat{g}(\hat{S}^{-k}\theta) + \sum_{k=1}^{\ell} H(F^{\ell-k}_{\hat{S}^{-\ell}\theta}(x))\,. \tag{6.2}$$

Applied to $x = \hat{\varphi}_{n-\ell}(\hat{S}^{-\ell}\theta)$ this yields

$$\log \hat{\varphi}_n(\theta) = \log \hat{\varphi}_{n-\ell}(\hat{S}^{-\ell}\theta) + \sum_{k=1}^{\ell} \log \hat{g}(\hat{S}^{-k}\theta) + \sum_{k=1}^{\ell} H(\hat{\varphi}_{n-k}(\hat{S}^{-k}\theta))\,. \tag{6.3}$$

If we apply the same reasoning to the system with multiplier $g \circ \Pi$, we get

$$\log \varphi_n(\Pi\theta) = \log \varphi_{n-\ell}(\Pi\hat{S}^{-\ell}\theta)) + \sum_{k=1}^{\ell} \log g(\Pi\hat{S}^{-k}\theta) + \sum_{k=1}^{\ell} H(\varphi_{n-k}(\Pi\hat{S}^{-k}\theta))\,. \tag{6.4}$$

As $\log \hat{g} = \log g \circ \Pi + \hat{b} - \hat{b} \circ \hat{S}^{-1}$ by (1.13), we can take the difference of (6.3) and (6.4) and obtain

$$\begin{aligned}\log \frac{\hat{\varphi}_n(\theta)}{\varphi_n(\Pi\theta)} &= \log \frac{\hat{\varphi}_{n-\ell}(\hat{S}^{-\ell}\theta)}{\varphi_{n-\ell}(\Pi\hat{S}^{-\ell}\theta)} + \hat{b}(\hat{S}^{-1}\theta) - \hat{b}(\hat{S}^{-\ell-1}\theta) \\ &\quad + \sum_{k=1}^{\ell}\Big(H(\hat{\varphi}_{n-k}(\hat{S}^{-k}\theta)) - H(\varphi_{n-k}(\Pi\hat{S}^{-k}\theta))\Big)\,.\end{aligned} \tag{6.5}$$

Let

$$\ell(n) = \min\Big\{k \in \{0,\dots,n\} : \hat{\varphi}_{n-k}(\hat{S}^{-k}\theta) \geqslant \varphi_{n-k}(\Pi\hat{S}^{-k}\theta)\Big\}\,. \tag{6.6}$$

The index $\ell(n)$ is well defined, because $\hat{\varphi}_0(\hat{S}^{-n}\theta) = a = \varphi_0(\Pi\hat{S}^{-n}\theta)$. We have $\hat{\varphi}_{n-k}(\hat{S}^{-k}\theta) < \varphi_{n-k}(\Pi\hat{S}^{-k}\theta)$ for $k = 1,\dots,\ell(n)-1$, and as $H' < 0$, we conclude from (6.5) that

$$\begin{aligned}\log \frac{\hat{\varphi}_n(\theta)}{\varphi_n(\Pi\theta)} &\geqslant -2\|\hat{b}\|_\infty + \Big(H(\hat{\varphi}_{n-\ell(n)}(\hat{S}^{-\ell(n)}\theta)) - H(\varphi_{n-\ell(n)}(\Pi\hat{S}^{-\ell(n)}\theta))\Big) \\ &\geqslant -2\|\hat{b}\|_\infty + H(a)\end{aligned} \tag{6.7}$$

provided $\ell(n) \geqslant 1$. If $\ell(n)0$, this estimate is trivially satisfied. Similarly one proves that $\log \frac{\hat{\varphi}_n(\theta)}{\varphi_n(\Pi\theta)} \leqslant 2\|\hat{b}\|_\infty - H(a)$. Therefore,

$$\left|\log \frac{\hat{\varphi}_n(\theta)}{\varphi_n(\Pi\theta)}\right| \leqslant 2\|\hat{b}\|_\infty + |H(a)|\,. \tag{6.8}$$

In the limit $n \to \infty$ we conclude that $\hat{\varphi}_\infty(\theta) > 0$ if and only if $\varphi_\infty(\Pi\theta) > 0$ and that $|\log \hat{\varphi}_\infty(\theta) - \log \varphi_\infty(\Pi\theta)| \leqslant 2\|\hat{b}\|_\infty + |H(a)|$ for such θ.

6.2. **The set of regular points and Remark 5a.** Recall that W_ϵ is the ϵ-neighbourhood of the finite set E of endpoints of monotonicity intervals of S and that μ denotes some S-invariant probability measure. We assume that there is some $q>0$ such that $\mu(W_\epsilon)=\mathcal{O}\left(\left(\log\log\frac{1}{\epsilon}\right)^{-(1+3q)}\right)$ as $\epsilon\to 0$, which is equivalent to (2.15).

Let $\tilde{n}_k:=\lfloor\exp(k^{\frac{1}{1+q}})\rfloor$, and observe that $d_k:=\tilde{n}_{k+1}-\tilde{n}_k\geqslant C\ \exp(k^{\frac{1}{1+2q}})$ for some $C>0$. Fix $r>0$ and suppose that, for some $v\in\mathbb{T}^1$, $S^nv\in W_r$ for all $n\in(\tilde{n}_k,\tilde{n}_{k+1}]$. As S is a piecewise expanding Markov map, $S(E)\subseteq E$ and $|(S^n)'|\geqslant C\lambda^n$ for some $C>0$ and $\lambda>1$. If $r>0$ is chosen sufficiently small, this implies that $S^{\tilde{n}_k}v\in W_{\lambda^{-d_k}}$. Hence,

$$\begin{aligned}\mu\left\{v\in\mathbb{T}^1:\ S^nv\in W_r \text{ for all } n\in(\tilde{n}_k,\tilde{n}_{k+1}]\right\}&\leqslant\mu\left(S^{-\tilde{n}_k}W_{\lambda^{-d_k}}\right)=\mu\left(W_{\lambda^{-d_k}}\right)\\ =\mathcal{O}\left(\left(\log\log\lambda^{d_k}\right)^{-(1+3q)}\right)&=\mathcal{O}\left((\log d_k)^{-(1+3q)}\right)=\mathcal{O}\left(k^{-\frac{1+3q}{1+2q}}\right)\ .\end{aligned}\tag{6.9}$$

Now the Borel-Cantelli Lemma implies that for μ-a.e. $v\in\mathbb{T}^1$ there is $k_v\in\mathbb{N}$ such that for all $k\geqslant k_v$ there is some $n_k\in(\tilde{n}_k,\tilde{n}_{k+1}]$ such that $S^{n_k}v\notin W_r$. These n_k satisfy

$$\limsup_{k\to\infty}\frac{n_{k+1}}{n_k}\leqslant\limsup_{k\to\infty}\frac{\tilde{n}_{k+2}}{\tilde{n}_k}\leqslant\limsup_{k\to\infty}\exp\left((k+2)^{\frac{1}{1+q}}-k^{\frac{1}{1+q}}\right)=1\ ,\tag{6.10}$$

and routine arguments for piecewise C^{1+} expanding Markov maps show the existence of a constant $\Delta>0$ (depending on r chosen above) such that (2.12) is satisfied.

6.3. **Anosov surface diffeomorphsims and their Markov maps.** Choose one fixed $\hat{S}^{-1}$-unstable fibre in each rectangle of the Markov partition $\{R_1,\dots,R_p\}$ and identify these p fibres isometrically with intervals $J_1,\dots,J_p$. Denote by J the disjoint union of $J_1,\dots,J_p$ and by $\varsigma:J\to\Theta=\mathbb{T}^2$ the map identifying the fibres and the intervals. Define $\Pi:\Theta\to J$ as the map that projects a point $\theta\in R_i$ along its $\hat{S}^{-1}$-stable fibre to the fibre $\varsigma(J_i)$ and then by ς^{-1} to J_i. Glueing the J_i at their endpoints turns J into a copy of $\mathbb{T}^1$ and affects only finitely many points in J.

Now we can define a map $S:\mathbb{T}^1\to\mathbb{T}^1$ by $S(v)=\Pi(\hat{S}^{-1}(\varsigma v))$. By construction, each $S(J_i)$ is a union of intervals J_j, and the resulting map is a Markov map w.r.t. the partition into intervals $J_i\cap S^{-1}J_j$. We must check that S is piecewise C^{1+}.

Recall from [13, eq. (8) in the proof of Lemma III.3.2] that Π, the holonomy map along $\hat{S}^{-1}$-stable fibres, is C^{1+} with derivative

$$D\Pi(\theta)=\lim_{N\to\infty}\left|\frac{D_u\hat{S}^{-N}(\theta)}{D_u\hat{S}^{-N}(\varsigma\Pi\theta)}\right|\ .\tag{6.11}$$

Observe that $\varsigma\Pi\hat{S}^{-1}\varsigma\Pi\theta = \varsigma\Pi\hat{S}^{-1}\theta$ by construction of Π and ς. Therefore

$$\begin{aligned}
\frac{D\Pi(\theta)}{D\Pi(\hat{S}^{-1}\theta)} &= \lim_{N\to\infty}\left|\frac{D_u\hat{S}^{-N}(\hat{S}^{-1}\theta)D_u\hat{S}^{-1}(\theta)}{D_u\hat{S}^{-N}(\hat{S}^{-1}\varsigma\Pi\theta)D_u\hat{S}^{-1}(\varsigma\Pi\theta)}\Big/\frac{D_u\hat{S}^{-N}(\hat{S}^{-1}\theta)}{D_u\hat{S}^{-N}(\varsigma\Pi\hat{S}^{-1}\theta)}\right| \\
&= \left|\frac{D_u\hat{S}^{-1}(\theta)}{D_u\hat{S}^{-1}(\varsigma\Pi\theta)}\Big/\lim_{N\to\infty}\frac{D_u\hat{S}^{-N}(\hat{S}^{-1}\varsigma\Pi\theta)}{D_u\hat{S}^{-N}(\varsigma\Pi\hat{S}^{-1}\varsigma\Pi\theta)}\right| \\
&= \left|\frac{D_u\hat{S}^{-1}(\theta)}{D_u\hat{S}^{-1}(\varsigma\Pi\theta)\cdot D\Pi(\hat{S}^{-1}\varsigma\Pi\theta)}\right| \\
&= \left|\frac{D_u\hat{S}^{-1}(\theta)}{S'(\Pi\theta)}\right| .
\end{aligned} \tag{6.12}$$

7. Large deviations for S

Piecewise expanding mixing C^{1+} Markov maps of $\mathbb{T}^1$ which are endowed with a positive α-Hölder continuous weight function g have the following property: There is some $\alpha' > 0$ (that depends on α and the minimal expansion of S) such that the transfer operator $\mathcal{L}_s$ introduced in (2.1) has a simple leading eigenvalue $\lambda_s > 0$ and

$$\mathcal{L}_s^n\xi = \lambda_s^n\zeta_s m_s(\xi) + O(\gamma_s^n) \tag{7.1}$$

for each function $\xi : \mathbb{T}^1 \to \mathbb{R}$ that is α'-Hölder restricted to each Markov interval of S. Here ζ_s is a strictly positive eigenfunction, m_s is a probability measure on $\mathbb{T}^1$ with full topological support, and $\gamma_s < \lambda_s$ [1, 14].

Suppose now that $(J_n)_{n\geqslant 1}$ is a sequence of subintervals of $\mathbb{T}^1$ with $\inf_n |J_n| > 0$. Fix $s \in \mathbb{R}$. Then $\inf_n m_s(J_n) > 0$, because otherwise one could find a subsequence (J_{n_i}) with $\lim_{i\to\infty} m_s(J_{n_i}) = 0$ and a nontrivial interval J that is contained in all these J_{n_i}. But then $m_s(J) = 0$ in contradiction to the fact that m_s has full support. It follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\frac{\int_{J_n} e^{-s\log g_n}\,dm}{m(J_n)} &= \lim_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{T}^1}\mathcal{L}_s^n 1_{J_n}\,dm \\
&= \lim_{n\to\infty}\frac{1}{n}\log\left(\lambda_s^n m_s(J_n)\int_{\mathbb{T}^1}\zeta_s\,dm + O(\gamma_s^n)\right) \\
&= \log\lambda_s = \log\rho(\mathcal{L}_s) = \psi(s)\,,
\end{aligned}$$

and this is a smooth strictly convex function of s. So we are in the situation to apply the large deviations theorem of Plachky/Steinebach [15], and this yields the estimate in (4.10).

References

[1] V. Baladi. *Positive Transfer Operators and Decay of Correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific, 2000.

[2] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. Springer, 2nd edition, 2007.

[3] X. Bressaud and C. Liverani. Anosov diffeomorphisms and coupling. *Ergodic Theory and Dynamical Systems*, 22:129–152, 2002.

[4] U. Feudel, S. Kuznetsov, and A. Pikovsky. *Strange Nonchaotic Attractors - Dynamics between Order and Chaos in Quasiperiodically Forced Systems*, volume 56 of *Series on Nonlinear Science, Series A*. World Scientific, 2006.

[5] P. W. Glynn and W. Whitt. Logarithmic asymptotics for steady-state tail probabilities in a single-server queue. *Journal of Applied Probability*, 31:131–156, 1994.
[6] T. Jäger. *The Creation of Strange Non-Chaotic Attractors in Non-Smooth Saddle-Node Bifurcations*, volume 945 of *Memoirs of the American Mathematical Society*. 2009.
[7] G. Keller. A note on strange nonchaotic attractors. *Fundamenta Mathematica*, 151:139–148, 1996.
[8] G. Keller, H. Jaffri, and R. Ramaswamy. Generalized synchronozation in chaotically driven systems. *In preparation*, 2012.
[9] G. Keller and A. Otani. Bifurcation and Hausdorff dimension in families of chaotically driven maps with multiplicative forcing. *Preprint arXiv:1208.2888*, 2012. See also the M. Sc. Thesis of A. Otani (Erlangen, 2011).
[10] M. Lelarge. Tail asymptotics for discrete event systems. *Discrete Event Dynamical Systems*, 18:563–584, 2008.
[11] D. V. Lindley. The theory of queues with a single server. *Journal of the Cambridge Philosophical Society*, 48:277–289, 1952.
[12] R. M. Loynes. The stability of a queue with non-independent inter-arrival and service times. *Proc. Cambridge Philos. Soc*, 58:497–520, 1962.
[13] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, 1987.
[14] W. Parry and M. Pollicott. *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, volume 187-188 of *Astérisque*. Société Mathématique de France, 1990.
[15] D. Plachky and J. Steinebach. A theorem about probabilities of large deviations with an application to queuing theory. *Periodica Mathematica Hungarica*, 6:343–345, 1975.
[16] O. Podvigina and P. Ashwin. On local attraction properties and a stability index for heteroclinic connections. *Nonlinearity*, 24:887–929, Mar. 2011.
[17] M. Pollicott. Hausdorff dimension and asymptotic cycles. *Transactions of the American Mathematical Society*, 355:3241–3252, 2003.
[18] C. Robinson. *Dynamical Systems*. CRC Press, 1995.
[19] T. Singh, A. Nandi, and R. Ramaswamy. Scenarios for generalized synchronization with chaotic driving. *Physical Review E*, 78(2):1–4, Aug. 2008.
[20] R. Sturman and J. Stark. Semi-uniform ergodic theorems and applications to forced systems. *Nonlinearity*, 13:113–143, 2000.

Department Mathematik, Universität Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany
E-mail address: keller@mi.uni-erlangen.de